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A SEMIPRIME MORITA CONTEXT RELATED TO FINITE AUTOMORPHISM GROUPS OF RINGS

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ABSTRACT

A Morita context relating the fixed ring and the skew group ring introduced by M. Cohen is studied. If the skew group ring is semiprime and R^G satisfied a PI, then R satisfies a PI of degree $\leqq |G|d$. We also discuss the Galois correspondence for the maximal quotient ring of a free algebra.

Introduction and preliminaries

Let R be a ring with identity and G a finite group of automorphisms of R. Let $R^G = \{r \in R \mid r^g = r \text{ for all } g \in G\}$, called the fixed ring. We study a Morita context between R^G and the skew group ring $S = R*G$. This context was introduced by M. Cohen [1]. There she examines the situation of S semiprime which implies that the trace is nondegenerate and R is semiprime. There are two important cases of when S is semiprime: Let R be semiprime (1) if R has no $|G|$ torsion by Fisher–Montgomery [14, theorem 7.4], (2) G is X-outer Montgomery [14, theorem 3.17].

In this note, we continue the study of the assumption that S is semiprime. The idea is to study group actions of semiprime nonsingular rings by passing to the skew group ring of the quotient ring, then returning to $R*G$ and finally to R. This is similar to the plan in Goursaud, Osterburg, Pascaud and Valette [5]. The difference is that in [5], we assumed the algebra of the group is semiprime, which is a more general assumption; however, the proofs to follow are considerably easier.

We begin this note by proving if $R*G$ is von Neumann regular then R contains an element of trace one and R^G is a "conner" ring of $R*G$. We then study semiprime nonsingular rings R and their quotient rings Q, proving among

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other things that Q is G-Galois over Q^G, if G is X-outer. We also give an example of a commutative regular ring and an X-outer group of order 2 such that R is not f.g. over R^G. Continuing an idea of Lorenz–Passman [9] we prove if R^*G is semisimple Artinian then R is f.g. over R^G and a bound on the number of generators is also given.

Then we show if R^*G is semiprime and R^G satisfies a PI of degree d, then R satisfies a PI of degree $\leqq |G|d$. V. K. Kharchenko showed this result if R has no $|G|$ torsion [14, theorem 6.5]. The other important time when R^*G is semiprime, namely G is X-outer, is a special case of [5, theorem C]. In fact, Goursaud et al. have shown if R and the algebra of the group are semiprime and R^G satisfies a PI of degree d, then R satisfies a PI of degree $\leqq |G|d$.

We finish the paper by studying group actions on free algebras. Kharchenko has shown in [6] that every such action is X-outer. Using our previous results, we know the maximal quotient ring Q is G-Galois over its fixed ring. We then apply our result that the fixed ring of the quotient ring is the quotient ring of the fixed ring and known results about the Galois correspondence to prove if G acts homogeneously and T is an intermediate subring. Then the following are equivalent:

(1) $T = Q^H$ for some subgroup H of G.

(2) T is a maximal quotient ring of an intermediate free subalgebra of R.

We begin with some definitions mainly to explain the notation. We have tried to follow the definitions and notation in [14]. Let $S = R^*G$ be the skew group ring, i.e. the free R module with basis G, $gr = r^{g^{-1}}g$ for $r \in R$, $g \in G$. We let $\operatorname{tr} r = \Sigma_G r^g$, $r \in R$, the trace.

Lemma 1. *If S is von Neumann regular then*: (1) *There exists a $d \in R$ with* $\operatorname{tr} d = 1$. (2) *Let $t = \Sigma_G g \in S$. Then $f = td$ is an idempotent and $\phi : R^G \to fR^*Gf$ via $x \to fxf$ is a ring isomorphism.*

Proof. Since R^*G is regular we pick $u = \Sigma_G u_g g \in R^*G$, s.t. $t = tut = \operatorname{tr}(\Sigma_G u_g)t$, where $\operatorname{tr} x = \Sigma_G x^g$, $x \in R$. Let $d = (\Sigma u_g)$. It can be checked if $f = td$ then $f^2 = f$ and $\phi : R^G \to fR^*Gf$, $\phi(x) = fxf = xf$ is a ring isomorphism.

Let $Q = Q_{\max} R$ be the left maximal quotient ring.

Corollary 1.1. *Let S be semiprime and R left nonsingular. Then R is semiprime and Q^G and Q^*G are regular selfinjective.*

Proof. R is semiprime by [1, theorem 1.21]. The group ring proof [18] goes over to the skew group ring situation, thus Q^*G is left selfinjective. Let

$u = \Sigma_G q_g g \in Q^*G$ s.t. $uQ^*Gu = 0$. Find an essential G-invariant left ideal D of R s.t. $Dq_g \subset R$. Thus $Du \subset R^*G$ and $Du(R^*G)\, Du \subseteq D(uQ^*Gu) = 0$. So $Du = 0$, thus $u = 0$ and we have proven Q^*G is semiprime.

Now by [18, theorem 7.2.5] the Jacobson radical of Q^*G is nilpotent, hence 0. Thus Q^*G is von Neumann regular [4, theorem 1.22], by Lemma 1 and [4, prop. 9.8] Q^G is regular selfinjective.

Following [12], we say R is *G-Galois over R^G* if there are $r_1, \cdots, r_k$; $r_1^*, \cdots, r_k^* \in R$ such that $\Sigma r_i r_i^{*g} = \delta_1, g$, where $\delta_1, g = 1$, if $g = 1$ and equals 0 if $g \neq 1$. We also insist there is a $d \in R$ with $\operatorname{tr} d = 1$. See also [1].

For the definition of X-outer, X-inner and G_{inn} see [14]. Consider the following statements: (i) Let Q be regular, left selfinjective and G X-outer, (ii) Q is prime regular left selfinjective and Q^*G is prime.

LEMMA 2. *If either* (i) *or* (ii) *is true, then Q is G-Galois over Q^G.*

PROOF. Every nonzero ideal of Q^*G intersects Q nonzeroly by [15, prop. 2.11] and [14, lemma 3.16]. Let M be a maximal two sided ideal of Q^*G. Using [4, cor. 9.15, theorem 8.20] we have $(M \cap Q)^*G$ is prime. For in (i) the center of Q^*G is in Q. Using Incomparability we have $M = (M \cap Q)^*G$ [8, theorem 1.2]. Let $t = \Sigma_G g$, by comparing coefficients, we see $t \notin M$ for any maximal ideal M of Q^*G. Thus $QtQ = Q^*G$. Hence Q is G-Galois over Q^G by [1, lemma 1.24].

In [1] the following Morita context is introduced and studied. Let $S = R^*G$ $s = \Sigma_G x_g g$, $r \in R$, let $\mathrm{sr} = \Sigma_G x_g r^{g^{-1}}$ and $\mathrm{rs} = \Sigma(rx_g)^g$. With these actions we have $V = {}_{R^G}R_S$ and $W = {}_S R_{R^G}$. Let

$$[\ \ ,\ \ ]: W \otimes V \to S, \quad [W, V] = wtv, \quad t = \sum_G g,$$

$$(\ \ ,\ \ ): V \otimes W \to R^G, \quad (v, w) = \operatorname{tr}(vw) = \sum_G (vw)^g.$$

Thus (R^G, V, W, S) is a Morita context.

COROLLARY 2.1. *Let R be semiprime and nonsingular, G-X-outer. Then* $[\ \ ,\ \ ]: W \otimes V \to S$ *is* $1-1$.

PROOF. Let Q = left maximal quotient ring of R. By Lemma 2 and [12, prop. 1.3), $[\ \ ,\ \ ]: Q \otimes Q \to Q^*G$ is $1-1$.

Next, we give a weak form of the normal basis theorem.

COROLLARY 2.2. *Let R be semiprime left and right Goldie, G-X-outer. Put $T = R^G G \subset R^*G$. Then T_T is isomorphic to a cyclic submodule of R, that is essential even as a right R^G submodule.*

PROOF. Let Q be the classical quotient ring of R. Then G is completely outer on Q [13, cor. 1.8] and [12, section 6]. By [16, prop. 3.2] Q has a left and right normal basis element q, i.e. $\{q^g \mid g \in G\}$ is a free basis of Q over Q^G. Now $q = a^{-1}b$, $a \in R^G$, $b \in R$ [14, theorem 5.3]. Let $x \in Q$, then $a^{-1}x = \Sigma a^{-1}b^{g_i}d_i$, $d_i \in Q^G$. Thus $\{b, b^{g_2}, \cdots, b^{g_n}\}$ is a free basis of Q over Q^G. Now let $r \in R$, $r = \Sigma b^{g_i}d_i$, $d_i \in Q^G$. Let $c \in Q^G$ be a common denominator, i.e. $d_i = c_i c^{-1}$, $c_i \in Q^G$ for each i. Thus we have for each $r \in R$ there exists $\{c, c_1, \cdots, c_n\} \subset R^G$ such that $(*)$ $rc = \Sigma_{i=1}^n b^{g_i}c_i$, $n = |G|$.

Let $\phi : T_T \to R_T$, $\phi(\Sigma r_i g_i) = b \cdot \Sigma r_i g_i = \Sigma b^{g_i} r_i$, $r_i \in R^G$. Now ϕ is additive and if $s \in R^G$, $h \in G$, $\phi(\Sigma r_i g_i sh) = \phi(\Sigma r_i s g_i h) = \Sigma b^{g_i h} r_i s = (\Sigma b^{g_i} r_i)sh$. Thus ϕ is a right T map. Finally $(*)$ shows Im ϕ is essential as a right R^G module.

In general, R does not have a normal basis. Let A be the Weyl algebra over the reals, i.e. generated by x, y with $yx - xy = 1$. Let h be the $\mathbf{R}$ automorphism of A such that $x^h = -x$, $y^h = -y$, $H = \{1, h\}$. By [1, theorem 1.27], A is H-Galois over A^H. Thus $A^*H = \mathrm{End}\, A_{A^H}$, if A had a normal basis over A^H, A^*H would be a 2×2 matric ring. But Zaleskii–Neroslavskii show this is impossible [21].

We give an example of a regular ring and a group of X outer automorphisms such that R is infinitely generated over R^G. Let F be the field of 2 elements and $F_n = F$, $n = 1,2,3,\cdots$. Consider the subring R of ΠF_n (the countable direct product) of sequences that eventually become constant. Let $g \in \mathrm{Aut}\ R$ that switches pairs, i.e.

$$g(f_1, f_2, \cdots, f_{2n-1}, f_{2n} \cdots) = (f_2, f_1, \cdots, f_{2n}, f_{2n-1}, \cdots).$$

$g^2 = 1$ and g is X-outer. Assume $\{r_1, \cdots, r_l\}$ is a basis of R over R^G. Then after a certain point, say l, each of them becomes constant. Let m be an odd integer $> l$. Consider $e_m \in R$ which has a 1 in the mth position and 0 everywhere else. Recall elements of the fixed ring are pairwise equal. So to achieve e_m as a combination of $r_1, \cdots, r_l$ we need to add an odd number of the r_1's to get the 1 in the mth position, but the r_1's are constant after l and the coefficients from the fixed ring have the same mth and $(m+1)$th components, so we must also have a 1 in the $(m+1)$th position.

Notice the maximal quotient ring of $R, Q = \Pi F_n$, is Galois over Q^G according to Lemma 3. The Galois basis is $a_1 = a_1^* = (1,0,1,0,1,0,\cdots)$, $a_2 = a_2^* = (0,1,0,1,0,1,\cdots)$. Also if $x \in Q$, $x = \mathrm{tr}(xa_1)a_1 + \mathrm{tr}(xa_2)a_2$, $\mathrm{tr}\, a_1 = 1$. So Q is f.g. over Q^G.

LEMMA 3. *Let T be a ring, G a finite subgroup of* Aut T *such that* T^*G *is* *semisimple Artinian. Then T is a* f.g. *projective T^G module. In fact, T_{T^G} can be generated with $\leqq |G| l(T)$ where $l(T) =$ length of T_T.*

PROOF. Let $A = \{u \in T^*G \mid u \cdot T = 0\}$, $U = T^*G/A$. U is semisimple Artinian and T is a faithful left U module. Let $U = L_1^{n_1} \oplus \cdots \oplus L_k^{n_k}$, where L_i's are the minimal pairwise nonisomorphic left ideals of U and n_i is the multiplicity of L_i. Since ${}_UT$ is faithful there are $r_1, \cdots, r_k$ elements of T s.t. $L_1 \approx L_i r_i \leqq T$. Since L_i is an injective U module, $L_i r_i$ is a direct summand of T. One can check $L_1 r_1 + \cdots + L_k r_k$ is direct. Let $n = \max[n_1, \cdots, n_k]$, then U is isomorphic to a direct summand of $L_1^n \oplus \cdots \oplus L_k^n \approx [L_1 \oplus \cdots \oplus L_k]^n =$ direct summand of T^n. Thus ${}_UT$ is a generator. Using Morita's theorem [2, theorem 4.1.3], we conclude T is a f.g. projective T^G module.

So T^G is semisimple Artinian [by lemma 1] and T_{T^G} has finite length. We now show $l(T_{T^G}) \leqq |G| l(T)$, where $l(\ )$ is length. Now we follow an idea of Lorenz–Passman [14, lemma 7.5]. Let V be a right T module and U a right T^G submodule of V. We form $W = V \otimes_T T^*G$. Each $w \in W$ can be uniquely written $w = \Sigma_G v_g \otimes g$, $v_g \in V$. Let $\mathrm{Tr}(w) = v_1$ and $U^\sigma = (U \otimes f)T^*G$. We note $fT^*Gt = T^G t$.

Thus $U^\sigma t = (U \otimes f)T^*Gt = U \otimes T^G t = U \otimes t$ as U is a right T^G module. Let $w^\tau = \mathrm{Tr}(wt)$, $w \in W$. Then $U^{\sigma\tau} = \mathrm{Tr}(U^\sigma t) = \mathrm{Tr}(U \otimes t) = U$. Thus if $U_{T^G} \leqq U^1_{T^G} \leqq V_{T_G}$ and $U^\sigma = U^{1\sigma}$, we have $U = U^1$.

Thus $l(T_{T^G}) \leqq l(T_{T^*G}) \leqq |G| l(T)$. Since T_{T^G} is completely reducible, we see that T_{T^G} can be generated by $\leqq |G|\ l(T)$ elements of T.

By L ess R we mean L is an essential left ideal of R.

LEMMA 4. *Let Q be regular left selfinjective such that Q^*G is semiprime. Then if $e^2 = e \in Q$ such that Qe (resp. eQ) is G-invariant, then there exists $f^2 = f \in Q^G$ such that $Qe = Qf$ (resp. $eQ = fQ$).*

PROOF. By Corollary 1.1, Q^G is regular selfinjective. So there is a direct summand D of Q^G s.t. $0 \neq \mathrm{tr}(Qe)$ ess D. Assume $D = Q^G f[f^2 = f \in Q^G]$. Now let $L = \mathrm{tr}(Qe) \oplus Q^G(1-f)$. One checks that L ess Q^G. Since $\mathrm{tr}(Qe)f = \mathrm{tr}(Qe)$, we have $\mathrm{tr}(Qe(1-f) = 0$. Thus $e = ef$ so $Qe \subset Qf$.

We now show QL ess Q. Since Q is selfinjective, there is a direct summand $Qu[u^2 = u \in Q]$ of Q s.t. QL ess Qu. Observe Qu/QL is the singular summand D of Q^G s.t. $0 \neq \mathrm{tr}(Qe)$ ess D. Assume $D = Q^G f[f^2 = f \in Q^G]$. Now Q/Qu is nonsingular, we have $(Qu)^g/QL$ is the singular submodule of Q/QL. Thus $Qu = (Qu)^g$.

So Qu is G-invariant, hence its annihilator $(1-u)Q$ is G-invariant.

Observe $L\,\mathrm{tr}[(1-u)Q]=0$, $L \operatorname{ess} Q^G$ and Q^G nonsingular. Thus $1=u$ or $QL \operatorname{ess} Q$. Now $QL \subseteq Qe + Q(1-f) \operatorname{ess} Q$. Because $e=ef$, we have $[Qe+Q(1-f)](fe-f)=0$. Thus $fe=f$ or $Qf \subseteq Qe$. Thus $Qe=Qf$.

Let $Z(\ \ )$ be the left singular submodule.

Lemma 5. *Let $R*G$ be semiprime, R^G left nonsingular. If L ess R^G, then RL ess R.*

Proof. As before, let Q be the left maximal quotient ring of R. Since $Z({}_RR)\cap R^G \subseteq Z(R^G)=0$, the semiprimeness of $R*G$ implies $Z(R)=0$. Thus Q is regular selfinjective. By Corollary 1.1, $Q*G$ is semiprime. So take $e^2=e \in Q$ s.t. QL ess Qe with Lemma 4 we assume $e \in Q^G$. Thus if $0\neq 1-e$, $Q(1-e)$ is G-invariant. Take K ess R, K a G-invariant left ideal such that $0\neq 1-e$, $Q(1-e)$ is G-invariant. Take K ess R, K a G-invariant left ideal such that $0\neq K(1-e)\subseteq R$. Now $(\mathrm{tr}\,K(1-e))\cap L=0$, which is a contradiction to L being essential. So $QL \operatorname{ess} Q$, by [3, prop. 218, p. 45] we have $RL \operatorname{ess} R$.

Theorem 6. *Let $R*G$ be semiprime. If either* (a) $Z({}_{R^G}R^G)$, (b) $Z({}_{R^G}R)$ *or* (c) $Z({}_RR)$ *is* 0, *then all are* 0. *Moreover if* (a), (b) *or* (c) *is* 0, *then* $[Q_{\max}R]^G = Q_{\max}(R^G)$ *where* $Q_{\max}(\ \ )$ *denotes the left maximal ring of quotients.*

Proof. By [1, cor. 1.5, theorem 1.21] (a) $=0$ implies (b) $=0$ and (c) $=0$. Lemma 5 can be employed to show (c) $=0$ implies (a) $=0$. Assume one (hence all) of (a), (b) or (c) is zero. Let $Q=Q_{\max}R$, by Lemma 1 and [3, prop. 2.11, p. 46] to show $Q^G=Q_{\max}(R^G)$ we must prove ${}_{R^G}R^G \operatorname{ess} {}_{R^G}Q^G$. But this is routine.

Lemma 7. *Let Q be regular selfinjective G a finite group of X-outer automorphisms of Q. Then ${}_{Q^G}Q$ is* f.g. *projective and injective.*

Proof. Lemma 2 and [4, theorem 9.2].

Theorem 8. *Let $R*G$ be semiprime such that R^G satisfies a* PI *of degree d. Then R satisfies a* PI *of degree $\leqq |G|d$.*

Proof. The singular ideal of R^G is zero and the maximal quotient ring of R^G satisfies the same PI [10, theorem 2]. Using Lemma 1 we assume without loss of generality $R*G$, R, and R^G are regular selfinjective.

We prove R has bounded index. It is enough to prove R/M is Artinian for every maximal 2-sided ideal M of R [4, theorem 7.20 and concluding remarks]. Let M be a maximal two sided ideal of R, $M'=\bigcap_G M^g$ $\bar R = R/M'$. Now $\bar R*G = R*G/M'*G$, a regular ring. Together with [8, lemma 4.1], we have $\bar R*G$

is a finite direct sum of simple rings. Recall from Lemma 1 there is a $d \in R$ with $\operatorname{tr} d = 1$. It is not hard to show $(\bar{R})^G = R^G / M' \cap R^G$ which is isomorphic to $f\bar{R}*Gf$, a direct sum of simple rings, where $f = \bar{d}t$, $\bar{d} = d + M'$. Since R^G satisfies a PI, we know by Kaplansky's theorem $\bar{R}^G$ is semisimple Artinian. Let $\bar{L} = L + M'$ be an essential left ideal of $\bar{R}$. Since $\bar{R}*G$ is regular, the trace of a nonzero G-invariant left ideal is nonzero [14, theorem 2.2]. Thus tr L is essential [14, lemma 5.1] forcing tr L to be $\bar{R}^G$. Thus $\bar{L} = \bar{R}$ and we have proven $\bar{R}$ is Artinian. Thus by [4, theorem 7.20] R has bounded index and primitive factor rings are Artinian. Using [4, cor. 6.16, cor. 9.27] we have the intersection of the maximal two-sided ideals is 0.

Let M be the maximal two sided ideal of R, $M' = \bigcap_G M^g$, $T = R/M'$, a semisimple artinian ring. Then $T^*G \cong R^*G/M'^*G$ is regular, so T^*G is semisimple Artinian. Using Lemma 3, we have T_{T^G} is f.g.

Thus T is a PI ring.

To finish the argument we estimate the degree of the PI, basically we modify [14, prop. 6.1]. Let $T = R/\cap M^g$, M a maximal ideal of R. Assume T is a simple ring f.d. over its center Z and T^*G is semiprime and G is inner. Then by [15, prop. 2.6], [17, lemma 2] the algebra of the group B is semisimple Artinian. Let $1 = \Sigma_{k=1}^n u_k$, where u_k is a central primitive idempotent of B. Let $T_j = u_j T u_j$, T_j is a simple ring f.d. over is center Zu_j. From [19, theorem 4, p. 111] it follows $T_j \otimes_{Zu_j} B^0_{u_j} \cong \operatorname{Mat}_{n_j} T^G$, where $n_j = (Bu_j : Zu_j)$. Note $\Sigma n_j = n = (B : Z)$ and T_j satisfies a PI of degree $\geqq nd$.

Now $R = \operatorname{Mat}_q D$, D a division ring $T_j \cong \operatorname{Mat}_{q_j} D_j$, $q = \Sigma q_j$. Let $(d : Z) = k^2$, so $(T_j : Z) = (kg_j)^2$. Thus it follows $2kq_j \leqq dn_i$. So $(t : Z) = (kq)^2 = \Sigma_i (kq_i)^2 \leqq \Sigma (d^2/4) n_i^2 \leqq (d^2/4) n^2 \leqq (\frac{1}{2} d \,|\, G \,|)^2$. So T satisfies $S_{d|G|}$. This handles the inner case.

By the Skolem Noether theorem [19] G/G_{inn} is outer on $T^{G_{\text{inn}}}$. Thus $T^{G_{\text{inn}}}$ can be embedded in $M_{|G|}(T^{G_{\text{inn}}})$ [14, theorem 2.7]. This finishes the care of T simple. We handle the general case exactly as in [14, prop. 6.1, p. 91]. Thus $R / \bigcap M'$ satisfies $S_{d|G|}$. Since $\bigcap M = 0$, we see R satisfies $S_{d|G|}$.

We conclude with some remarks concerning X-outer groups and G-Galois extensions. The most elegant case of a free algebra over a field was studied by Kharchenko in [6]. Combining his results with previous results we obtain the following result.

Let k be a field $R = k\langle x_1, \cdots, x_n \rangle$, the free algebra, G a finite group of automorphisms. Let Q be the maximal left quotient ring of R. Q is a simple regular selfinjective ring [20, corollaire 2.3]. We say T is an intermediate subring of R (resp. of Q), if $R^G \leqq T \leqq R$ resp. $Q^G \leqq T \leqq Q$).

THEOREM 9. *With the above notation the following are equivalent*: (1) $T = Q^H$ *for some subgroup H of G*, (2) *T is simple regular and selfinjective s.t.* $T/\mathbf{Q}$ *is nonsingular. Moreover if the elements of G are homogeneous then* (1) *is equivalent to* : (3) *T is the left maximal quotient ring of an intermediate free subalgebra of R*.

PROOF. It is known that the left maximal quotient ring of R is simple regular selfinjective, see [20].

G is X-outer by [7, prop. 2] or by [11, cor. 5C]. Thus G is completely outer on Q [14, example 3.6, 44]. Let T be a simple regular sefinjective intermediate subring of Q. By Lemma 7, ${}_{Q^G}Q$ is f.g., hence ${}_TQ$ is f.g. Since ${}_TQ$ is nonsingular by [4, theorem 9.2], we have ${}_TQ$ is projective and injective. By [12, theorem 6.10], $T = Q^H$ for some H.

Conversely if H is a subgroup of G, $T = Q^H$ is simple, because there is an element e such that $\mathrm{tr}_H e = \Sigma_H e^h = 1$ [14, theorem 2.5]. Let $G = \bigcup_{j=1}^k Hg_j$ recall from Lemma 2 there is a $d \in Q$ such that $\mathrm{tr}\, d = 1$. Let $e = \Sigma_{j=1}^k d^{g_j}$. It can be easily checked $\mathrm{tr}_H\, e = 1$. T is regular selfinjective by Corollary 1 and ${}_TQ$ is nonsingular by Theorem 6. Finally assume G is homogeneous. From [7] we have a $1-1$ correspondence between subgroups H of G and intermediate free subalgebras of R. Employing Theorem 6 we have our result.

THEOREM 10. *Let R be a prime left nonsingular ring, G X-outer, H a subgroup of G. Let* $F: R^H \to R$ *be an embedding, then F can be extended to an element of G.*

PROOF. Let Q be the maximal quotient ring of R. Use theorem to extend F to Q^H. Now quote [12, theorem 4.2]; note we need [14, cor. 6.17].

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